1.

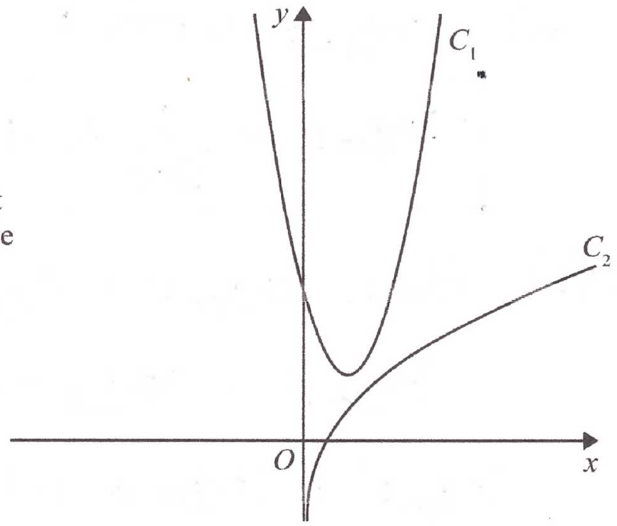

Figure 3

The curve C_1, shown in Figure 3, has equation $y = 4x^2 - 6x + 4$.

The point $P\left(\frac{1}{2}, 2\right)$ lies on C_1

The curve C_2, also shown in Figure 3, has equation $y = \frac{1}{2}x + \ln(2x)$.

The normal to C_1 at the point P meets C_2 at the point Q.

Find the exact coordinates of Q.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(8)

$y = 4x^2 - 6x + 4$

$\frac{dy}{dx} = 8x - 6$

As P lies on C_1, substitute in $x = \frac{1}{2}$ to find the gradient at P:

$\frac{dy}{dx} = 8\left(\frac{1}{2}\right) - 6$

$= 4 - 6$

$= -2$

Then, the gradient of the normal at P must be $\frac{1}{2}$

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Question continued

Equation of normal at P: $y - y_1 = m(x - x_1)$

Using $P(\frac{1}{2}, 2)$ and $m = \frac{1}{2}$ $\rightarrow$ $y - y_1 = m(x - x_1)$

$$y - 2 = \frac{1}{2}\left(x - \frac{1}{2}\right)$$

$$2y - 4 = x - \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{7}{4}$$

To find the point of intersection between the normal and C_2, solve simultaneous equations:

$y = \frac{1}{2}x + \frac{7}{4}$ (1) $\quad$ $y = \frac{1}{2}x + \ln(2x)$ (2)

Substitute (1) into (2): $\frac{1}{2}x + \frac{7}{4} = \frac{1}{2}x + \ln(2x)$

$$\ln(2x) = \frac{7}{4}$$

$$2x = e^{7/4}$$

$$x = \frac{e^{7/4}}{2}$$

Substitute into (1) for y: $y = \frac{1}{2}\left(\frac{e^{7/4}}{2}\right) + \frac{7}{4}$

$$y = \frac{e^{7/4}}{4} + \frac{7}{4}$$

$\therefore$ coordinates of Q are $\boxed{\left(\frac{e^{7/4}}{2}, \frac{e^{7/4} + 7}{4}\right)}$

(Total for Question 15 is 8 marks)

2. A curve has equation

$$y = 2x^3 - 4x + 5$$

Find the equation of the tangent to the curve at the point $P(2, 13)$.

Write your answer in the form $y = mx + c$, where m and c are integers to be found.

Solutions relying on calculator technology are not acceptable.

(5)

tangent : same gradient, same coordinate, one point of intersection $\leftrightarrow$ one root

differentiate $y(x)$: $y = 2x^3 - 4x + 5 \Rightarrow \frac{dy}{dx} = 3 \times 2x^{(3-1)} - 4x^{(1-0)}$

$= 6x^2 - 4$

so gradient @ $P = 6(2)^2 - 4 = 20$

use $y - y_0 = m(x - x_0)$: $y - 13 = 20(x-2)$

$y - 13 = 20x - 40$

$\underline{y = 20x - 27}$

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3. A circle C with centre at $(-2, 6)$ passes through the point $(10, 11)$.

(a) Show that the circle C also passes through the point $(10, 1)$.

(3)

The tangent to the circle C at the point $(10, 11)$ meets the y axis at the point P and the tangent to the circle C at the point $(10, 1)$ meets the y axis at the point Q.

(b) Show that the distance PQ is 58 explaining your method clearly.

(7)

(a) Radius of circle = distance between $(-2,6)$ and $(10,11)$

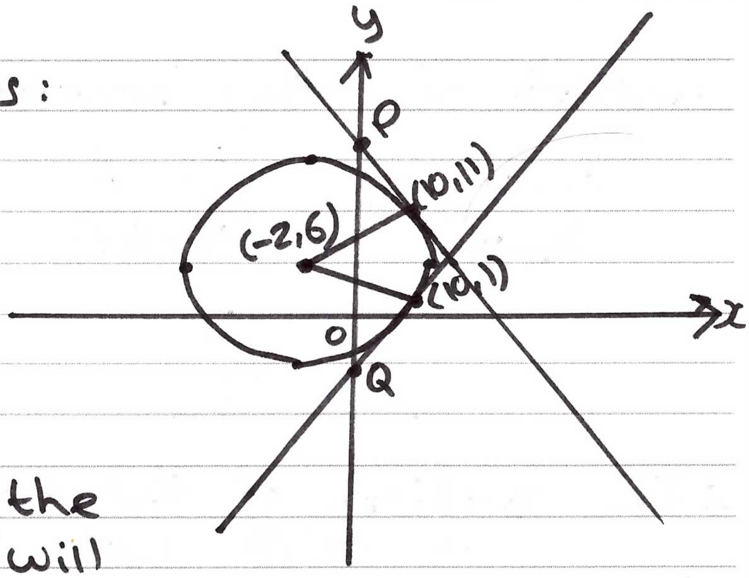
$$\text{Radius} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$= \sqrt{(-2-10)^2+(6-11)^2}$$
$$= \sqrt{(-12)^2+(-5)^2}$$
$$= \sqrt{144+25}$$
$$= \sqrt{169}$$
$$= 13 \text{ units}$$

Now, distance between $(-2,6)$ and $(10,1)$:

$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$= \sqrt{(-2-10)^2+(6-1)^2}$$
$$= \sqrt{(-12)^2+5^2}$$
$$= \sqrt{144+25}$$
$$= \sqrt{169}$$
$$= 13 \text{ units}$$

Distance are equal, and so $(10,1)$ lies on the circle

Question continued



(b) Gradient of radius:

$$\frac{y_2-y_1}{x_2-x_1} = \frac{11-6}{10-(-2)}$$

$$= \frac{5}{12}$$

$\therefore$ the gradient of the tangent at (10,11) will be $-\frac{12}{5}$

Equation of tangent at (10,11):

$$y-y_1 = m(x-x_1)$$

$$y-11 = -\frac{12}{5}(x-10)$$

$$5(y-11) = -12(x-10)$$

$$5y-55 = -12x+120$$

$$12x+5y-175=0$$

When this line cuts the y-axis, $x=0$

$$\therefore \quad 5y-175=0$$

$$5y=175 \Rightarrow y=35$$

$\therefore$ P is at $(0,35)$

Question continued

Gradient of radius between centre and (10,1):

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1-6}{10-(-2)}$$

$$= \frac{-5}{12}$$

$\therefore$ the gradient of the tangent at (10,1) will be $\frac{12}{5}$.

Equation of tangent at (10,1):

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{12}{5}(x - 10)$$

$$5(y-1) = 12(x-10)$$

$$5y - 5 = 12x - 120$$

$$12x - 5y - 115 = 0$$

When this line cuts the y-axis, $x = 0$

$\therefore -5y = 115 \Rightarrow y = -23$

$\therefore$ Q is at $(0, -23)$

Distance PQ $= 35 + 23$ $\boxed{= 58}$

(Total for Question is 10 marks)

TOTAL FOR PAPER IS 100 MARKS

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4. The curve C has equation

$$y = \frac{k^2}{x} + 1 \qquad x \in \mathbb{R},\ x \neq 0$$

where k is a constant.

(a) Sketch C stating the equation of the horizontal asymptote. **(3)**

The line l has equation $y = -2x + 5$

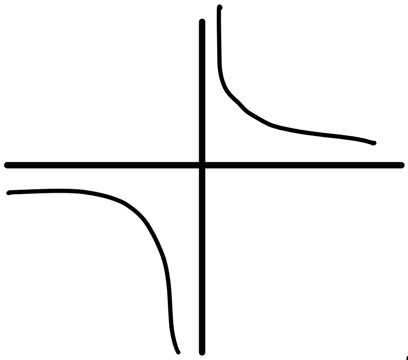
(b) Show that the x coordinate of any point of intersection of l with C is given by a solution of the equation

$$2x^2 - 4x + k^2 = 0$$

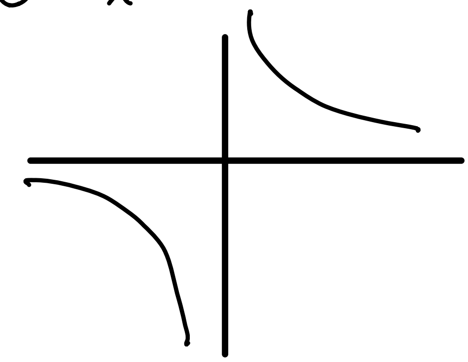
(2)

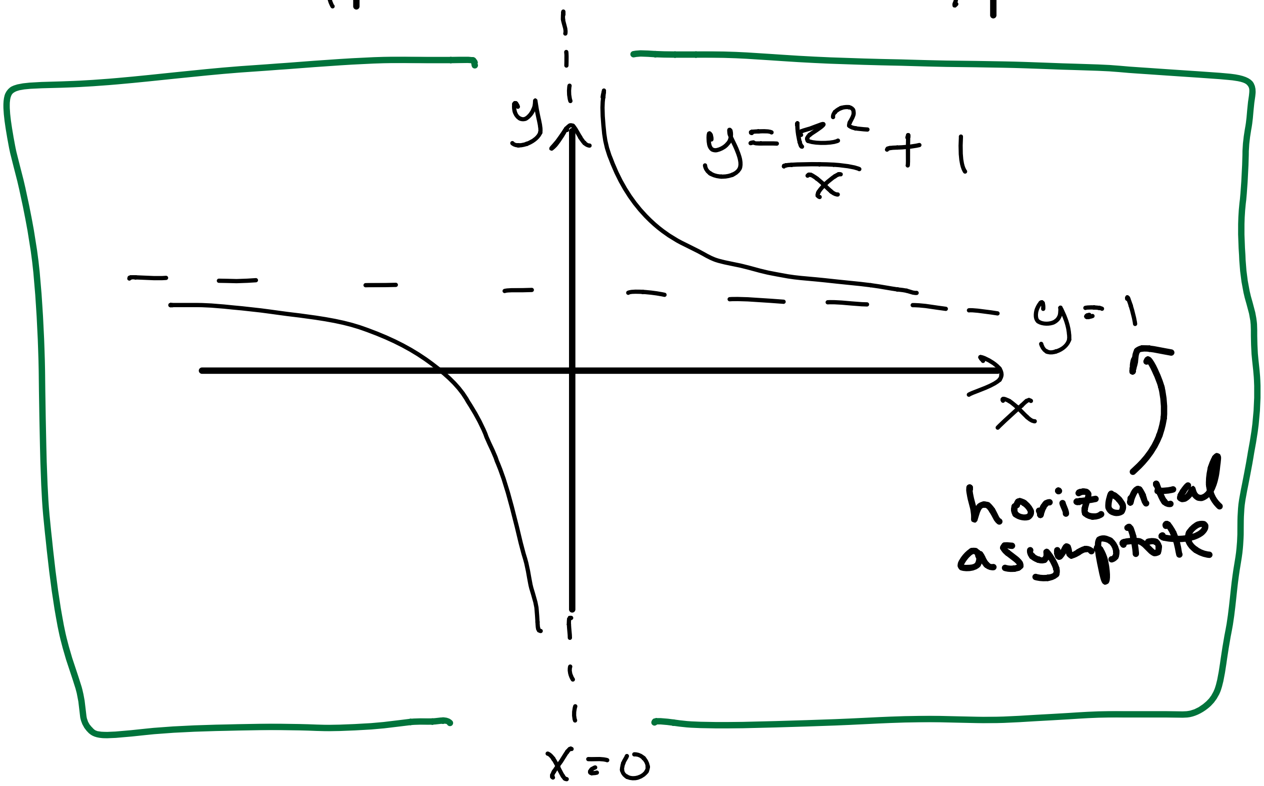
(c) Hence find the exact values of k for which l is a tangent to C. **(3)**

a) $y = 1/x$



$y = \frac{k^2}{x}$





Question continued

b) $y = \frac{k^2}{x} + 1$ and $y = -2x + 5$

$\Rightarrow \frac{k^2}{x} + 1 = -2x + 5$

$\Rightarrow k^2 + x = -2x^2 + 5x$

$\Rightarrow 2x^2 - 4x + k^2 = 0$ //

c) tangent; L will only meet C at one point.

So $b^2 - 4ac = 0$

$\Rightarrow (-4)^2 - 4(2)(k^2) = 0$

$\Rightarrow 16 = 8k^2$

$\Rightarrow 2 = k^2$

$\Rightarrow \boxed{k = \pm\sqrt{2}}$

(Total for Question is 8 marks)

5.

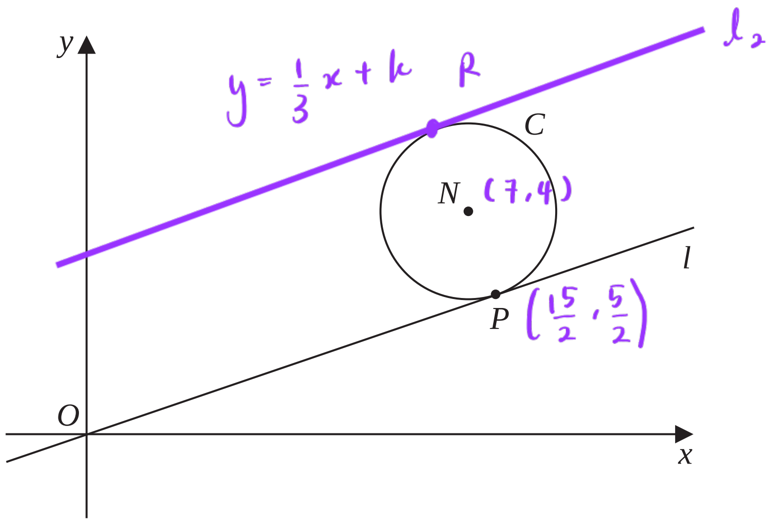


Figure 4

Figure 4 shows a sketch of a circle C with centre $N(7, 4)$

The line l with equation $y = \frac{1}{3}x$ is a tangent to C at the point P.

Find

(a) the equation of line PN in the form $y = mx + c$, where m and c are constants,

(2)

(b) an equation for C.

(4)

The line with equation $y = \frac{1}{3}x + k$, where k is a non-zero constant, is also a tangent to C.

(c) Find the value of k.

(3)

(a) line l has equation $y = \frac{1}{3}x$. Hence, the gradient is $\frac{1}{3}$

gradient of PN = $\frac{-1}{1/3} = -3$

Use coordinates of N (7, 4) to form the equation.

PN : $y - (4) = -3(x - 7)$ 1

PN : $y - 4 = -3x + 21$

PN : $y = -3x + 25$ 1

Question 5 continued

(b) Find coordinates of P using line l and PN

$$\frac{1}{3}x = -3x + 25 \quad (1)$$

$$\frac{1}{3}x + 3x = 25$$

$$\frac{10}{3}x = 25$$

$$x = \frac{25 \times 3}{10}$$

$$x = 7.5$$

$$x = \frac{15}{2}$$

$$y = \frac{1}{3} \times \frac{15}{2}$$ → substitute x into $y = \frac{1}{3}x$ to find the y coordinate

$$= \frac{15}{6}$$

$$= \frac{5}{2}$$

$$\therefore P\left(\frac{15}{2}, \frac{5}{2}\right) \quad (1)$$

$$r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$r^2 = \left(\frac{15}{2} - 7\right)^2 + \left(\frac{5}{2} - 4\right)^2$$

$$= \left(\frac{1}{2}\right)^2 + \left(-\frac{3}{2}\right)^2$$

$$= \frac{1}{4} + \frac{9}{4}$$

$$= \frac{10}{4}$$

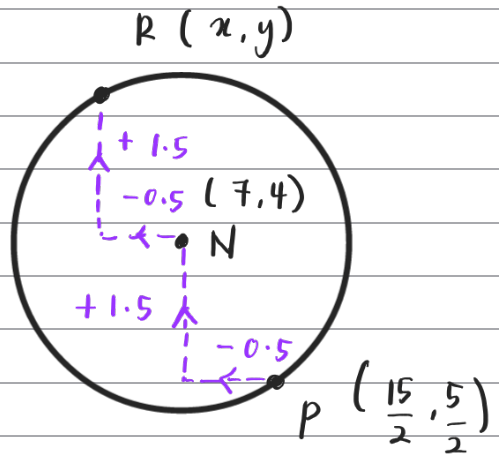
$$r^2 = \frac{5}{2}$$

$$r = \sqrt{\frac{5}{2}} \quad (1)$$

Question 5 continued

equation for C : $(x-7)^2 + (y-4)^2 = \frac{5}{2}$ # ①

(c)



Coordinates of R : $((7-0.5), (4+1.5))$

R : $(6.5, 5.5)$

R : $\left(\frac{13}{2}, \frac{11}{2}\right)$ ①

Given $y = \frac{1}{3}x + k$ ← substitute coordinate of R into this

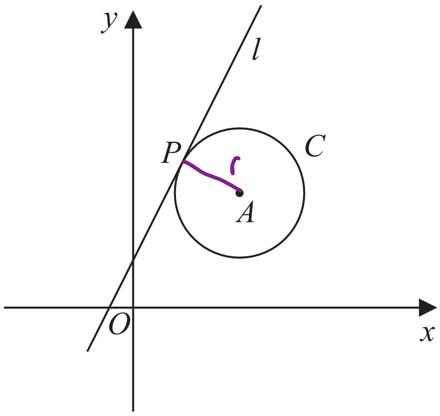
$\frac{11}{2} = \frac{1}{3}\left(\frac{13}{2}\right) + k$ ①

$\frac{11}{2} = \frac{13}{6} + k$

$k = \frac{11}{2} - \frac{13}{6}$

$k = \frac{10}{3}$ # ①

6.



Not to scale

$r =$ radius of C

Figure 3

The circle C has centre A with coordinates $(7, 5)$.

The line l, with equation $y = 2x + 1$, is the tangent to C at the point P, as shown in Figure 3.

$\hookrightarrow m_t = 2$

(a) Show that an equation of the line PA is $2y + x = 17$ **(3)**

(b) Find an equation for C. **(4)**

The line with equation $y = 2x + k$, $k \neq 1$ is also a tangent to C.

(c) Find the value of the constant k. **(3)**

a) m_t = tangent gradient. m_r = radius gradient.

for perpendicular lines, $m_1 m_2 = -1$

$m_t \times m_r = -1$

$2 \times m_r = -1$

$m_r = -\frac{1}{2}$ ✓

$y - y_1 = m(x - x_1)$

(x_1, y_1) is a point on the line

$x_1 = 7 \quad y_1 = 5$

$y - 5 = -\frac{1}{2}(x - 7)$ ✓

$2y - 10 = -(x - 7)$

$2y - 10 = -x + 7$

$2y + x = 17$ as required. ✓

Question continued

b)

$PA: 2y + x = 17 \quad l: y = 2x+1 \quad A(7,5)$

$(x-a)^2 + (y-b)^2 = r^2$

$(x-7)^2 + (y-5)^2 = r^2$

$2(2x+1) + x = 17$

$4x + 2 + x = 17$

$5x + 2 = 17$ ✓

$5x = 15 \quad \therefore x = 3 \quad \Rightarrow y = 2(3)+1$

$= 6+1$

$= 7.$

$P = (3, 7)$ ✓

$|PA| = \sqrt{(P_x - A_x)^2 + (P_y - A_y)^2}$

$= \sqrt{(3-7)^2 + (7-5)^2} = \sqrt{16+4} = \sqrt{20}$ ✓

$r = \sqrt{20} \quad \therefore r^2 = 20$

Equation of C is $(x-7)^2 + (y-5)^2 = 20$ ✓

Question continued

c)

$C: (x-7)^2 + (y-5)^2 = 20 \qquad y = 2x+k$

tangent => solution exist.

$C: x^2 - 14x + 49 + y^2 - 10y + 25 = 20$

$x^2 - 14x + y^2 - 10y + 54 = 0$

$x^2 - 14x + (2x+k)^2 - 10(2x+k) + 54 = 0$

$x^2 - 14x + 4x^2 + 4kx + k^2 - 20x - 10k + 54 = 0$

$5x^2 + (4k-34)x + k^2 - 10k + 54 = 0$

$ax^2 + bx + c$

tangent => $\underline{\text{one}}$ solution only : $b^2 - 4ac = 0$

$(4k-34)^2 - \underbrace{4(5)}_{-20}(k^2 - 10k + 54) = 0$

$16k^2 - 272k + 1156 - 20k^2 + 200k - 1080 = 0$

$-4k^2 - 72k + 76 = 0$

$k^2 + 18k - 19 = 0$

$(k+19)(k-1) = 0$

$k + 19 = 0 \Rightarrow k = -19$

$k - 1 = 0 \Rightarrow k = 1$

$k = -19$ & 1, but since $k \neq 1$, $\therefore k = -19$